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AN INVESTIGATION OF APPROXIMATE SOLUTIONS TO FLUID DYNAMICS AND HEAT TRANSFER PROBLEMS BY VARIATIONAL METHODS

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ABSTRACT

Approximate solutions for fluid dynamics and thermal problems by variational method were investigated. The application of the variational technique, based on a least square procedure originally suggested by Citron, was illustrated by several specific examples. Solutions were obtained for a melting-freezing problem, a natural convection problem, and two transient heat condition problems. Comparison with existing exact solutions were made to demonstrate the degree of accuracy of the method used.

Nomenclature

c, c_p	-	Specific heat
E	-	Squared error functional defined by Eq. (1.4)
g	-	Gravitational acceleration
Gr	-	Grashof Number
H	-	Generalized function, defined by Eq. (4.13)
h	-	Convection heat transfer coefficient
k	-	Thermal conductivity
L	-	Reference length or latent heat of fusion
Nu	-	Nusselt Number
p	-	Pressure
p_n	-	Generalized coordinates
Pr	-	Prandtl Number
Q	-	Generalized function, defined by Eq. (1.2)
q_n	-	Generalized coordinates
Ra	-	Rayleigh Number
T	-	Temperature
t	-	Time variable
U	-	Dimensionless velocity component in the x direction
u	-	Velocity component in the x direction
V	-	Dimensionless velocity component in the y direction
v	-	Velocity component in the y direction
X	-	Dimensionless space variable
x, y	-	space variables
α	-	Thermal diffusivity
β	-	Coefficient of thermal expansion

δ_o	-	Depth of freezing front at time t
δ_1, δ_2	-	Thermal penetration thickness
θ	-	Dimensionless temperature variable
μ	-	Dynamic viscosity
ν	-	Kinematic viscosity
ξ	-	Dimensionless space variable in the x direction
ρ	-	Mass density
τ	-	Time variable, or dimensionless time variable
$\bar{\psi}$	-	Stream function
ψ	-	Dimensionless stream function
ψ_M	-	Hydrodynamic penetration thickness defined in Section 4
ψ_T	-	Thermal penetration thickness defined in Section 4

I. INTRODUCTION

The desirability of good approximate solution techniques for mathematically difficult problems need not be argued. Until relatively recently, approximate solutions for fluid dynamics and thermal problems were almost exclusively obtained by integral techniques. In recent years approximate solutions of the Galerkin type and the Biot type, founded on the calculus of variations, have begun to be used.

Most of the applications of Biot-type variational solutions have been to a limited class of conduction problems. The purpose of this report is to demonstrate the utility of Biot-type variational techniques for a wider variety of problems. Specific examples included in this report include a melting-freezing solution, a natural convection problem and two special conduction problems.

When Biot[1,2,3] originally proposed his variational technique, it was based on the ideas of minimum entropy production. Since then, it has been demonstrated in a number of ways that the same final results can be obtained on purely mathematical grounds. These later expositions have the advantage of removing the thermodynamic limitation implicit in Biot's original work, and allow the procedure to be viewed as one that works for any physical phenomena governed by a particular type of partial differential equation. Of the various developments possible, a development based on an idea originally suggested by Citron[4] is favored in this report.

Some specific differential equation is almost required in a discussion of the variational technique. The transient heat conduction example is chosen here, but as the examples contained later in this report demonstrate, the technique is not limited to this relatively simple type of problem.

Consider solutions to

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \tag{1.1}$$

for a region $x_1 \leq x \leq x_2$, $t > 0$ and subject to appropriate boundary and initial conditions. A new function Q is defined, where for a true solution

$$Q(x,t) = -\int_0^t k \frac{\partial T}{\partial x}\, dt \tag{1.2}$$

which yields with the differential equation

$$\frac{\partial Q}{\partial x} = -\int_0^t \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) dt = -\int_0^t \rho c \frac{\partial T}{\partial t}\, dt$$

$$\frac{\partial Q}{\partial x} = -\int_{T(x,o)}^{T(x,t)} \rho c dT \tag{1.3}$$

The functional E is defined as

$$E(t) = \int_0^t \int_{x_1}^{x_2} \left[\frac{\partial Q}{\partial t} + k \frac{\partial T}{\partial x}\right]^2 dx\, dt \tag{1.4}$$

For a true solution, $E(t) \equiv 0$ as a consequence of Equation (1.2). In constructing approximate solutions the procedure is to compute Q for a specified T through Equation (1.3), (not Equation (1.2)) and then to minimize the error represented by the functional defined by Equation (1.4).

Assume then that

$$T(x,t) = T(x,q_i,\tau) \tag{1.5}$$

where q_i represents n unspecified functions of time, i.e., $q_i = q_i(t)$, and $\tau = t$. It is assumed that $T(x,q_i,\tau)$ satisfies all boundary conditions required of the function T, but it does not necessarily satisfy boundary conditions imposed on space derivatives of the function T. As a consequence of Equation (1.3)

$$\frac{\partial Q}{\partial x} = Q_x(x,q_i,\tau) \tag{1.6}$$

and therefore, after integration,

$$Q(x,t) = Q(x,q_i,\tau) \tag{1.7}$$

Thus,

$$\frac{\partial Q}{\partial t} = \sum_{j=1}^{n} \frac{\partial Q}{\partial q_j} \dot{q}_j + \frac{\partial Q}{\partial \tau} \tag{1.8}$$

and

$$k \frac{\partial T}{\partial x} = k\, T_x(x,q_i,\tau) \tag{1.9}$$

Combining Equation (1.8) and Equation (1.9), it is found that

$$\frac{\partial Q}{\partial t} + k \frac{\partial T}{\partial x} = \sum_{j=1}^{n} \frac{\partial Q}{\partial q_j} \dot{q}_j + \frac{\partial Q}{\partial \tau} + k \frac{\partial T}{\partial x} \quad ,$$

when considered as a function of $(x,q_i,\dot{q}_i,\tau)$, is linear in the $\dot{q}_i$.

If we now consider the minimization of $E(t)$ as a result of variations of $\partial Q/\partial t$ but not $\partial T/\partial x$, or T, Equations (1.5) through (1.9) require that this is equivalent to a minimization with respect to the functions $\mathring{q}_i$ while the q_j are considered fixed. Application of Euler's equation then yields the n equations

$$\int_{x_1}^{x_2} \left[\frac{\partial Q}{\partial t} + k\,\frac{\partial T}{\partial x}\right] \frac{\partial Q}{\partial q_j}\, dx = 0 \quad , \quad j = 1,2,\ldots, n$$

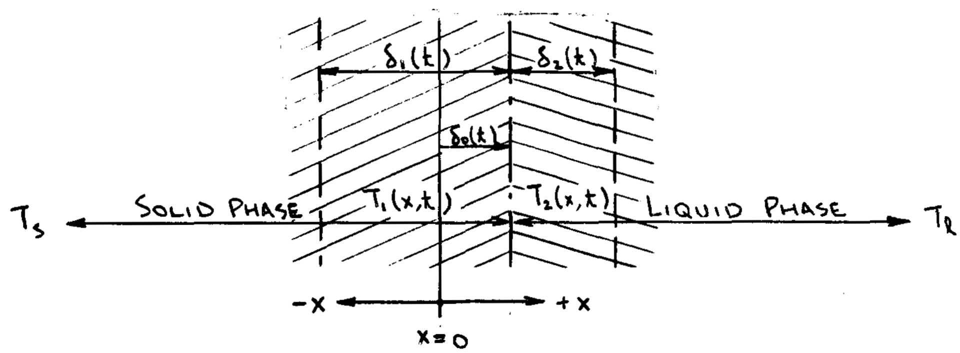
which are 1st order ordinary differential equations for the $q_i(t)$.

In the sections which follow, this variational procedure is applied to a number of different problems. The object of these sections is to illustrate the variety of particular techniques required and thus to suggest approaches for other problems. An additional object is to compare variational results with exact results so as to provide some insight to the accuracy of approximate solutions.

II. A Melting-Freezing Problem

Approximate variational solutions for melting-freezing problems are relatively easy to obtain, and appear to be highly accurate. As an example of the techniques involved, one specific problem is considered in this section, and the variational results are compared to results available from an exact solution(5).

The specific example considered is that of a semi-infinite solid in contact with a semi-infinite liquid of the same density.



The geometry is as illustrated in the sketch above. The subscript 1 shall refer to the left-hand, or solid, phase while the subscript 2 shall refer to the other. Assuming solidification is occurring, the interface between the two phases moves to the right with velocity $\dot{\delta}_0(t)$, where $\delta_0(t)$ is the distance from $x = 0$ to the interface.

The governing differential equations, assuming constant properties, are

$$\frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} = \frac{\partial^2 T_1}{\partial x^2} \qquad -\infty < x \leq \delta_0 \tag{2-1}$$

$$\frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} = \frac{\partial^2 T_2}{\partial x^2} \qquad \delta_0 \leq x < \infty$$

The initial conditons assumed for the problem are

$$T_1(x,0) = T_s, \text{ a constant} \qquad x < 0$$
$$T_2(x,0) = T_\ell, \text{ a constant} \qquad x > 0 \tag{2-2}$$

while the boundary conditions are

$$T_1(\delta_o,t) = T_2(\delta_o,t) = T_m, \text{ a constant}, \quad t > 0 \tag{2-3}$$

and

$$-k_1 \left.\frac{\partial T_1}{\partial x}\right)_{x=\delta_o} + \rho L \dot{\delta}_o = -k_2 \left.\frac{\partial T_2}{\partial x}\right)_{x=\delta_o} \tag{2-4}$$

where L represents the enthalpy difference between the saturated liquid and solid state. It has been assumed that the density of the liquid and solid state are identical.

In the solution of this problem it is assumed that the thermal effect in both the solid and liquid can be represented adequately by a penetration thickness. The two penetration thicknesses, $\delta_1(t)$ and $\delta_2(t)$, thus become the unknown functions sought by the variational procedure. These penetration thicknesses, as illustrated in the earlier sketch, are measured from the moving interface.

To formulate the variational problem, the following functions are defined.

$$Q_1 = \int_0^t -k_1 \frac{\partial T_1}{\partial x}\, dt \qquad -\infty < x \leq \delta_o$$
$$Q_2 = \int_0^t -k_2 \frac{\partial T_2}{\partial x}\, dt \qquad \delta_o \leq x < \infty \tag{2-5}$$

$$E_1 = \int_0^t \int_{-(\delta_1-\delta_0)}^{0} \left[\frac{\partial Q_1}{\partial t} + k_1 \frac{\partial T_1}{\partial x}\right]^2 dx\ dt \tag{2-6}$$

$$E_2 = \int_0^t \int_{\delta_0}^{\delta_0+\delta_2} \left[\frac{\partial Q_2}{\partial t} + k_2 \frac{\partial T_2}{\partial x}\right]^2 dx\ dt$$

The result of the variational procedure may be written as

$$\int_{-(\delta_1-\delta_0)}^{\delta_0} \frac{\partial Q_1}{\partial \delta_1}\left[\frac{\partial Q_1}{\partial t} + k_1 \frac{\partial T_1}{\partial x}\right] dx = 0 \tag{2-7}$$

$$\int_{\delta_0}^{\delta_0+\delta_2} \frac{\partial Q_2}{\partial \delta_2}\left[\frac{\partial Q_2}{\partial t} + k_2 \frac{\partial T_2}{\partial x}\right] dx = 0$$

From equations (2-1) and (2-5)

$$\frac{\partial Q_1}{\partial x} = \int_0^t \frac{\partial}{\partial x}\left(-k_1 \frac{\partial T_1}{\partial x}\right) dt = \frac{-k_1}{\alpha_1}\int_0^t \frac{\partial T_1}{\partial t}\, dt = -\rho c_1 (T_1 - T_s) \tag{2-8}$$

$$\frac{\partial Q_2}{\partial x} = -\rho c_2 (T_2 - T_\ell)$$

The temperature profiles used for this problem are

$$\frac{T_1 - T_s}{T_m - T_s} = (1 + \xi_1)^2 \qquad -1 \leq \xi_1 \leq 0 \tag{2-9}$$

$$\frac{T_2 - T_\ell}{T_m - T_\ell} = (1 - \xi_2)^2 \qquad 0 \leq \xi_2 \leq 1$$

where

$$\xi_1 = \frac{x-\delta_o}{\delta_1}$$

$$\xi_2 = \frac{x-\delta_o}{\delta_2} \qquad (2\text{-}10)$$

The temperature profiles chosen satisfy the boundary conditions of equation (2-3) as well as

$$T_1(-(\delta_1-\delta_o),\ t) = T_s$$

$$T_2(\delta_c+\delta_2,\ t) = T_\ell \qquad (2\text{-}11)$$

$$\left.\frac{\partial T_1}{\partial x}\right)_{x=-(\delta_1-\delta_o)} = 0 = \left.\frac{\partial T_2}{\partial x}\right)_{x=\delta_o+\delta_2} \qquad (2\text{-}12)$$

From equations (2-8) and (2-9) there results

$$Q_1 = -\frac{1}{3} A_1 \delta_1 (1+\xi_1)^3 \qquad -1 \le \xi_1 \le 0$$

$$Q_2 = \frac{1}{3} A_2 \delta_2 (1-\xi_2)^3 \qquad 0 \le \xi_2 \le 1 \qquad (2\text{-}13)$$

where

$$A_1 = \rho c_1(T_m-T_s)$$

$$A_2 = \rho c_2(T_m-T_\ell)$$

and in which the conditions

$$\left.Q_1\right)_{x=-(\delta_1-\delta_o)} = 0 = \left.Q_2\right)_{x=\delta_o+\delta_2} \qquad (2\text{-}14)$$

have been introduced. The conditions of equation (2-14) result from the definition of Q, equation (2-5), and the idea that beyond the penetration thickness, the heat flux is zero.

From equation (2-13) it follows that

$$\frac{\partial Q_1}{\partial t} = -\frac{1}{3} A_1 (1+\xi_1)^2 \left[\dot{\delta}_1 (1-2\xi_1) - 3\dot{\delta}_o \right]$$
$$\frac{\partial Q_2}{\partial t} = \frac{1}{3} A_2 (1-\xi_2)^2 \left[\dot{\delta}_2 (1+2\xi_2) + 3\dot{\delta}_o \right] \tag{2-15}$$

The boundary condition of equation (2-4) may be written as

$$\left.\frac{\partial Q_1}{\partial t}\right)_{\xi_1 = 0} + \rho L \dot{\delta}_o = \left.\frac{\partial Q_2}{\partial t}\right)_{\xi_2 = 0} \tag{2-16}$$

From equations (2-15) and (2-16) there results

$$\dot{\delta}_o = \frac{1}{3A} (A_1 \dot{\delta}_1 + A_2 \dot{\delta}_2) \tag{2-17}$$

where

$$A = (A_1 - A_2) + \rho L$$

Integrating equation (2-17) and noting that $\delta_o(0) = \delta_1(0) = \delta_2(0) = 0$ yields

$$\delta_o = \frac{1}{3A} (A_1 \delta_1 + A_2 \delta_2) \tag{2-18}$$

With the two previous expressions it is possible to eliminate δ_o and $\dot{\delta}_o$ from the expressions for Q and $\partial Q/\partial t$ and to compute

$$\frac{\partial Q_1}{\partial \delta_1} = -\frac{1}{3} A_1 (1 + \xi_1)^2 \left(1-2\xi_1 - \frac{A_1}{A}\right)$$

$$\frac{\partial Q_2}{\partial \delta_2} = \frac{1}{3} A_2 (1-\xi_2)^2 \left(1 + 2\xi_2 + \frac{A_2}{A}\right) \qquad (2\text{-}19)$$

Also, from equation (2-9)

$$k_1 \frac{\partial T_1}{\partial x} = 2A_1 \frac{\alpha_1}{\delta_1} (1 + \xi_1)$$

$$k_2 \frac{\partial T_2}{\partial x} = 2A_2 \frac{\alpha_2}{\delta_2} (1-\xi_2) \qquad (2\text{-}20)$$

Thus, if the results from equations (2-15), (2-17), (2-18), (2-19) and (2-20) are substituted into equations (2-7) and the indicated integrations are performed, the following ordinary differential equations result.

$$C_1 \dot{\delta}_1 + C_2 \dot{\delta}_2 - C_3 \frac{1}{\delta_1} = 0$$

$$D_1 \dot{\delta}_2 + D_2 \dot{\delta}_1 - D_3 \frac{1}{\delta_2} = 0 \qquad (2\text{-}21)$$

where

$$C_1 = \frac{13}{7} - \frac{8}{3}\frac{A_1}{A} + \left(\frac{A_1}{A}\right)^2 , \quad D_1 = \frac{13}{7} + \frac{8}{3}\frac{A_2}{A} + \left(\frac{A_2}{A}\right)^2$$

$$C_2 = -\frac{A_2}{A}\left(\frac{4}{3} - \frac{A_1}{A}\right) , \quad D_2 = \frac{A_1}{A}\left(\frac{4}{3} + \frac{A_2}{A}\right)$$

$$C_3 = \frac{\alpha_1}{2}\left(21 - 15\frac{A_1}{A}\right) , \quad D_3 = \frac{\alpha_2}{2}\left(21 + 15\frac{A_2}{A}\right)$$

The solution to equation (2-21) which satisfies the physical conditions of the problem and the initial condition $\delta_1(0) = 0 = \delta_2(0)$ may be written

$$\delta_1(t) = \beta\, \delta_2(t) \tag{2-22}$$

$$\delta_2(t) = \left[\frac{2\, D_3}{D_1 + D_2\, \beta}\right]^{1/2} (\sqrt{t}\,) \tag{2-23}$$

where

$$\beta = \frac{1}{2\, C_1 D_3} \left[(C_3 D_2 - C_2 D_3) + \sqrt{(C_3 D_2 - C_2 D_3)^2 + 4\, C_1 C_3 D_1 D_3}\right]$$

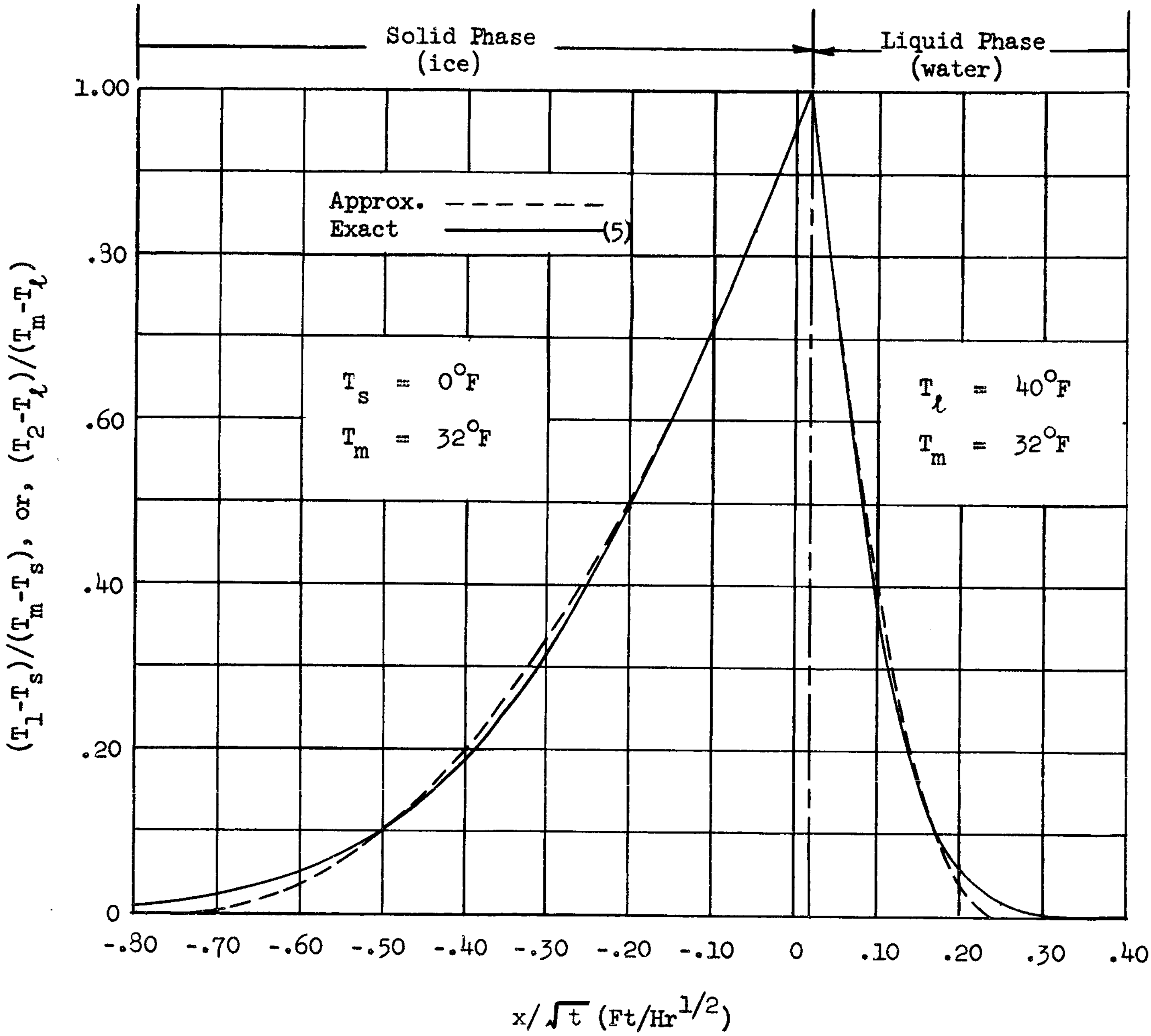
The solution obtained has been compared with the available exact solution for the specific case of freezing water. The conditions chosen were $T_m = 32^\circ F$, $T_s = 0^\circ F$, $T_\ell = 40^\circ F$, $\rho_2 = \rho_1 = 57\,{}^{lbm}/ft^3$ and $L = 143\ {}^{Btu}/lb_m$. The exact solution yields

$$\delta_o(t) = 0.0186\, \sqrt{t} \tag{2-24}$$

while the approximate solution yields

$$\delta_o(t) = 0.0188\, \sqrt{t} \tag{2-25}$$

The temperature profiles for the approximate and exact solutions are shown on the following graph. It is obvious that the agreement is outstandingly good.


Solid Phase
(ice)
Liquid Phase
(water)
Approx.
Exact
(5)
T_s = 0°F
T_m = 32°F
T_ℓ = 40°F
T_m = 32°F
1.00
.80
.60
.40
.20
0
-.80
-.70
-.60
-.50
-.40
-.30
-.20
-.10
0
.10
.20
.30
.40
$(T_1-T_s)/(T_m-T_s)$, or, $(T_2-T_\ell)/(T_m-T_\ell)$
$x/\sqrt{t}$ (Ft/Hr$^{1/2}$)

III. Transient Conduction with non-uniform initial conditions

Previous solutions for transient conduction problems by variational methods have been limited to problems involving uniform initial conditions. In this section, variational solutions for two problems with non-uniform initial conditions are obtained.

For one space variable transient conduction with constant properties, the governing differential equation is

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{3.1}$$

Introducing the following dimensionless variables

$$\theta = \frac{T - T_1}{T_2 - T_1} \quad , \quad \tau = \frac{kt}{\rho c L^2} \quad , \quad X = \frac{x}{L}$$

in which T_1 and T_2 are suitable reference temperatures and L is a suitable reference length, equation (3.1) may be written as

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} \tag{3.2}$$

In order to implement the variational procedure, a new variable $Q(X,\tau)$ is defined by

$$Q(X,\tau) = -\int_0^{\tau} \frac{\partial \theta}{\partial X} \, d\tau \tag{3.3}$$

It is apparent that $Q(X,\tau)$ is proportional to the heat flux over the time τ.

In the usual way, the definition of

$$E(\tau) = \int_0^{\tau} \int_{X_1}^{X_2} \left[\frac{\partial Q}{\partial \tau} + \frac{\partial \theta}{\partial X}\right]^2 dX \, d\tau \tag{3.4}$$

and the specification that the set of unknown functions involved in the variational procedure are designated by $q_n(\tau)$, leads to the following equation as the variational result for this problem.

$$\int_{X_1}^{X_2} \frac{\partial Q}{\partial q_n} \left[\frac{\partial Q}{\partial \tau} + \frac{\partial \theta}{\partial X}\right] dX = 0 \tag{3.5}$$

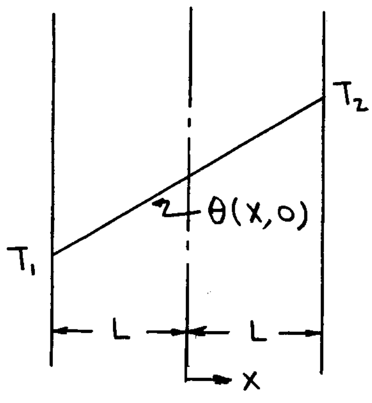
The function Q is computed from an assumed temperature profile by noting that from equations (3.2) and (3.3) it follows that

$$\begin{aligned} \frac{\partial Q}{\partial X} &= -\int_0^\tau \frac{\partial^2 \theta}{\partial X^2}\, d\tau = -\int_0^\tau \frac{\partial \theta}{\partial \tau}\, d\tau \\ &= \theta(X,o) - \theta(X,\tau) \end{aligned} \tag{3.6}$$

Therefore Q may be determined by integration of equation (3.6).

Problem I

The first problem considered is that of a slab, of thickness 2L, with an initial linear temperature distribution, subjected to perfect insulation of both surfaces for time greater than zero. In the adjacent figure the reference temperatures and the location of the coordinate axis are illustrated. The initial condition may be written



$$\theta(X,0) = \frac{1}{2}(1 + X) \tag{3.7}$$

The perfect insulation boundary condition for $\tau > 0$ requires

$$Q(1,\tau) = Q(-1,\tau) = 0 \tag{3.8}$$

Equation (3.8) follows from the interpretation of Q given with equation (3.3).

Since the variation of θ about the mid-plane value of $\frac{1}{2}$ is an odd function of X, θ is assumed to be of the form

$$\theta = \frac{1}{2} + q_1(\tau)X + q_2(\tau)X^3 \tag{3.9}$$

Additional terms involving higher odd powers of X could be taken, but it is expected that the solution obtained from equation (3.9) is adequate for almost all purposes. From the initial condition, equation (3.7), it follows that

$$q_1(0) = \frac{1}{2} \quad , \quad q_2(0) = 0 \tag{3.10}$$

Utilizing equations (3.6) and (3.9)

$$\frac{\partial Q}{\partial X} = \left[\frac{1}{2} - q_1(\tau)\right]X - q_2(\tau)X^3 \tag{3.11}$$

Thus

$$Q = \frac{1}{2}\left[\frac{1}{2} - q_1(\tau)\right](X^2-1) - \frac{1}{4}\,q_2(\tau)(X^4-1) \tag{3.12}$$

where the condition $Q(1,\tau) = 0$ has been used.

It is now possible to compute the following terms required for equation (3.5)

$$\frac{\partial Q}{\partial \tau} = \frac{(1-X^2)}{2}\,\dot{q}_1 + \frac{(1-X^4)}{4}\,\dot{q}_2 \tag{3.13}$$

$$\frac{\partial Q}{\partial q_1} = \frac{(1-X^2)}{2} \tag{3.14}$$

$$\frac{\partial Q}{\partial q_2} = \frac{1-X^4}{4} \tag{3.15}$$

$$\frac{\partial \theta}{\partial X} = q_1 + 3q_2\,X^2 \tag{3.16}$$

Introducing the previous expressions into equation (3.5) yields the following two ordinary differential equations for q_1 and q_2.

$$\int_0^1 \left(\frac{1-X^2}{2}\right)\left[\frac{1-X^2}{2}\,\dot{q}_1 + \frac{1-X^4}{4}\,\dot{q}_2 + q_1 + 3q_2X^2\right]dX = 0 \tag{3.17}$$

$$\int_0^1 \left(\frac{1-X^4}{4}\right)\left[\frac{1-X^2}{2}\,\dot{q}_1 + \frac{1-X^4}{4}\,\dot{q}_2 + q_1 + 3q_2X^2\right]dX = 0 \tag{3.18}$$

Performing the indicated integrations reduces these equations to

$$14\,\dot{q}_1 + 8\,\dot{q}_2 + 35\,q_1 + 21\,q_2 = 0 \tag{3.19}$$

$$24\,\dot{q}_1 + 14\,\dot{q}_2 + 63\,q_1 + 45\,q_2 = 0 \tag{3.20}$$

The solution to these equations is given by

$$q_1(\tau) = A_1e^{-\lambda_1\tau} + A_2e^{-\lambda_2\tau} \tag{3.21}$$

$$q_2(\tau) = B_1e^{-\lambda_1\tau} + B_2e^{-\lambda_2\tau} \tag{3.22}$$

with $$\lambda_i = 14 \pm \sqrt{133} \tag{3.23}$$

and $$B_i = \frac{-A_i}{33}\,(y + 2\,\lambda_i) \tag{3.24}$$

The initial conditions, equation (3.10), yields

$$\begin{aligned} A_1 + A_2 &= \frac{1}{2} \\ B_1 + B_2 &= 0 \end{aligned} \tag{3.25}$$

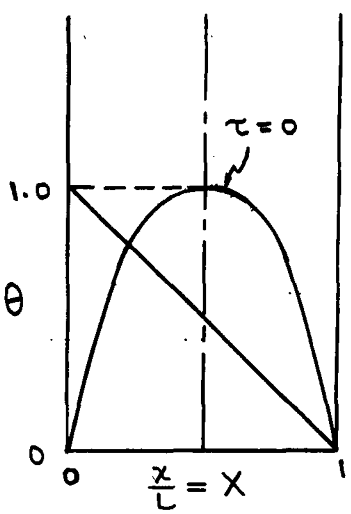
Computed results are given in the following table along with values from the exact solution. The agreement is excellent.

Time $\tau = \frac{kt}{\rho c L^2}$	Position $\frac{x}{L} = X = 0$		$\frac{x}{L} = X = 1/2$		$\frac{x}{L} = X = 1$	
	Exact[5]	Variational	Exact	Variational	Exact	Variational
0.0	.500	.500	.750	.750	1.000	1.000
0.2	.500	.500	.676	.675	.749	.746
0.4	.500	.500	.608	.607	.653	.650
0.8	.500	.500	.540	.540	.557	.556
1.6	.500	.500	.506	.506	.508	.508
∞	.500	.500	.500	.500	.500	.500

Values of $\theta = \frac{T-T_1}{T_2-T_1}$ for Problem I.

Problem II

The second problem considered is that of a slab, of thickness L, with an initial temperature distribution which is parabolic. For time greater than zero, the slab is subjected to a time dependent temperature on one surface while the other surface temperature remains constant. The adjacant figure illustrates the problem. The initial condition is



$$\theta\ (X,0) \ = \ 4\ X\ (1-X) \tag{3.26}$$

The boundary conditions are

$$\theta(1,\tau) \ = \ 0 \tag{3.27}$$

$$\theta(0,\tau) \ = \ 1 - e^{-100\tau} \tag{3.28}$$

The particular form chosen for $\theta(0,\tau)$ was dictated by a desire to have a function which allowed very rapid surface temperature changes yet was simple to work with.

The temperature, θ, is assumed to be of the form of a finite power series. The number of terms in the series is taken as five, but the procedure for fewer or more terms is similar. Thus

$$\theta(X,\tau) = q_o(\tau) + q_1(\tau)X + q_2(\tau)X^2 + q_3(\tau)X^3 + q_4(\tau)X^4 \tag{3.29}$$

Consideration of the boundary and initial conditions, equations (3.26)-(3.28) yields

$$\theta(X,\tau) = q_o(1-X) + q_2(X^2-X) + q_3(X^3-X) + q_4(X^4-X) \tag{3.30}$$

with

$$q_o(\tau) \ = \ 1-e^{-100\tau}$$

$$q_2(0) \ = \ -4$$

$$q_3(0) \ = \ 0 \ = \ q_4(0)$$

Equation (3.30) contains three unknown functions, q_2, q_3, and q_4, which must be determined from three differential equations.

Following the procedure indicated by equation (3.6), there results

$$Q = -q_o(X - \frac{X^2}{2}) + (q_2 + 4)(\frac{X^2}{2} - \frac{X^3}{3}) + q_3 (\frac{X^2}{2} - \frac{X^4}{4}) + q_4 (\frac{X^2}{2} - \frac{X^5}{5}) + q_5 \tag{3.31}$$

where $\quad q_5 = q_5(\tau) \equiv Q(0,\tau)$

It is noted that the absence of a heat flux boundary condition introduces an additional unknown function, $q_5(\tau)$. Since this unknown function, $q_5(\tau)$, does not appear in the expression for Θ, $q_5(\tau)$ appears in the variational results only through its first derivative, $\dot{q}_5(\tau)$. Additionally, the particular differential equation added through $q_5(\tau)$ is of a very simple form, and allows for an easy elimination of $q_5(\tau)$ from the other differential equations.

It is now possible to compute the following terms required for equation (3.5).

$$\frac{\partial Q}{\partial \tau} = (\frac{X^2}{2} - X)\dot{q}_o + (\frac{X^2}{2} - \frac{X^3}{3})\dot{q}_2 + (\frac{X^2}{2} - \frac{X^4}{4})\dot{q}_3 + (\frac{X^2}{2} - \frac{X^5}{5})\dot{q}_4 + \dot{q}_5 \tag{3.32}$$

$$\frac{\partial Q}{\partial q_2} = \frac{X^2}{2} - \frac{X^3}{3} \tag{3.33}$$

$$\frac{\partial Q}{\partial q_3} = \frac{X^2}{2} - \frac{X^4}{4} \tag{3.34}$$

$$\frac{\partial Q}{\partial q_4} = \frac{X^2}{2} - \frac{X^5}{5} \tag{3.35}$$

$$\frac{\partial Q}{\partial q_5} = 1 \tag{3.36}$$

$$\frac{\partial \Theta}{\partial X} = -q_o - q_2(1-2X) - q_3(1-3X^2) - q_4(1-4X^3) \tag{3.37}$$

Introducing the previous expressions into equation (3.5) and performing the indicated integrations yields the following ordinary differential equations.

$$312\dot{q}_2 + 447\dot{q}_3 + 518\dot{q}_4 + 2520\dot{q}_5 + 1008q_2 + 1512q_3 + 1800q_4 = 1092\dot{q}_o + 2520q_o \tag{3.38}$$

$$745\dot{q}_2 + 1070\dot{q}_3 + 1242\dot{q}_4 + 5880\dot{q}_5 + 2520q_2 + 3840q_3 + 4620q_4 = 2580\dot{q}_o + 5880q_o \tag{3.39}$$

$$\begin{aligned} &14{,}245\dot{q}_2 + 20{,}493\dot{q}_3 + 23{,}814\dot{q}_4 + 110{,}880\dot{q}_5 \\ &\quad + 49{,}500q_2 + 76{,}230q_3 + 92{,}400q_4 = 49{,}005\dot{q}_o + 110{,}880q_o \end{aligned} \tag{3.40}$$

$$5\dot{q}_2 + 7\dot{q}_3 + 8\dot{q}_4 + 60\dot{q}_5 = 20\dot{q}_o + 60q_o \tag{3.41}$$

As suggested before, the function q_5 appears only through its derivative $\dot{q}_5$. Thus equation (3.41) may be used to eliminate $\dot{q}_5$ from equations (3.38)-(3.40). The resulting equations may be solved in a straightforward manner. The solution may be written

$$q_2(\tau) = A_1e^{-\lambda_1\tau} + A_2e^{-\lambda_2\tau} + A_3e^{-\lambda_3\tau} + A_4e^{-100\tau} \tag{3.42}$$

$$q_3(\tau) = B_1e^{-\lambda_1\tau} + B_2e^{-\lambda_2\tau} + B_3e^{-\lambda_3\tau} + B_4e^{-100\tau} \tag{3.43}$$

$$q_4(\tau) = C_1e^{-\lambda_1\tau} + C_2e^{-\lambda_2\tau} + C_3e^{-\lambda_3\tau} + C_4e^{-100\tau} \tag{3.44}$$

where $\lambda_1 = \frac{6}{13}\left[111 - \sqrt{8031}\right]$

$\lambda_2 = 40$

$\lambda_3 = \frac{6}{13}\left[111 + \sqrt{8031}\right]$

$A_1 = 0.15061$, $A_2 = \frac{50}{3}$, $A_3 = 345.2734$, $A_4 = -366.0906$

$B_1 = -2.3242$, $B_2 = \frac{-100}{9}$, $B_3 = -567.9522$, $B_4 = 581.3876$

$C_1 = 1.1621$, $C_2 = 0$, $C_3 = 283.9761$, $C_4 = -285.1382$

This problem was also done by utilizing one less unknown function, i.e., by terminating the assumed series for Θ, equation (3.29), with the fourth term. The details of this simpler solution are not given here but the results for both solutions are compared to exact solution results in the table and graph that follow. It is to be noted that both solutions give an excellent representation of the temperature for long times. The representation for short times is 'good' for the higher order solution considering the severe test represented by this example.

X	$\tau = 0.01$			$\tau = 0.1$		
	Exact(5)	two unknown Functions	three unknown Functions	Exact	two unknown Functions	three unknown Functions
0	.632	.632	.632	1.000	1.000	1.000
.125	.512	.644	.548	.914	.917	.913
.250	.693	.707	.671	.825	.827	.826
.375	.859	.784	.835	.730	.727	.731
.500	.919	.834	.924	.621	.615	.622
.625	.860	.820	.873	.494	.489	.493
.750	.670	.702	.669	.346	.346	.345
.875	.364	.442	.349	.178	.184	.180
1.000	.000	.000	.000	.000	.000	.000

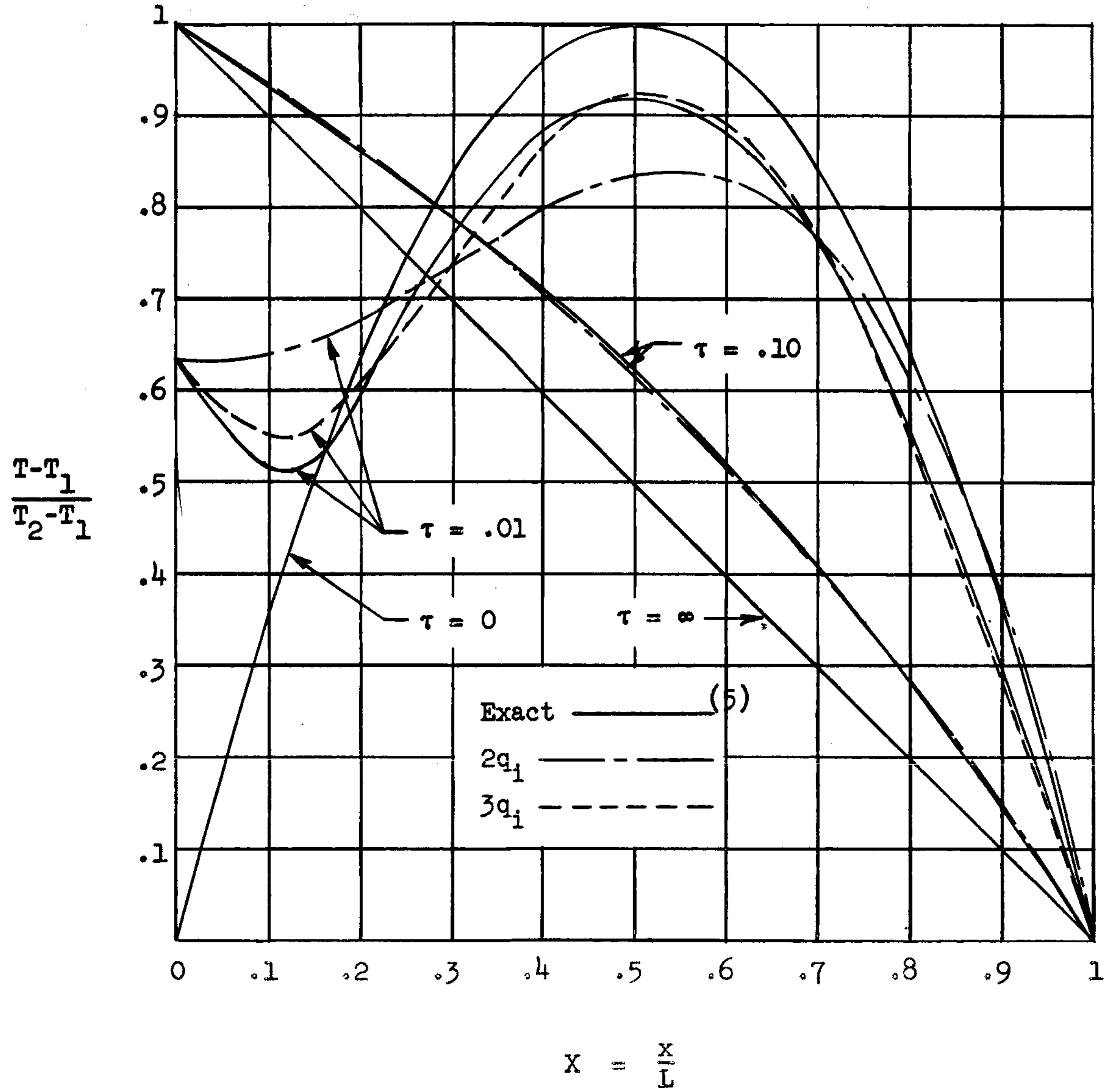


Figure II. Comparison of Exact and Variational solutions for a slab with parabolic initial temperature distribution and time dependent surface temperature.

IV. A NATURAL CONVECTION PROBLEM

A. Variational Equations

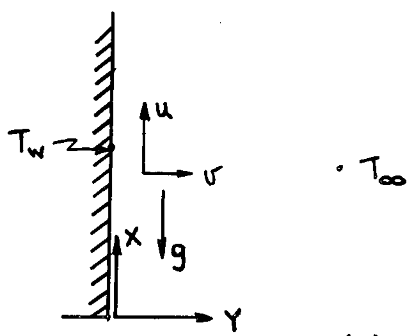
Steady laminar natural convection from a vertical semi-infinite flat plate is a problem which has been investigated extensively. There exist approximate integral solutions[6] and exact theoretical results[7] with which to compare a variational solution.

Introducing the standard boundary layer assumptions[8] into the steady state Navier-Stokes, energy and continuity equations yield

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{4.1}$$

$$\rho\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{dp}{dx} - \rho g \tag{4.2}$$

$$\rho C_p\left\{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right\} = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{dp}{dx} \tag{4.3}$$



As has been shown by Ostrach[7], it is permissible to neglect the dissipation and compression work terms in the energy equation if

$$\left|\beta(T - T_\infty)\right|$$

is small, as it almost always is. If, additionally, constant properties are assumed, and the purely hydrostatic pressure variation for y is introduced, the previous equations reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \beta g\,(T-T_\infty) \tag{4.4}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{4.5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.6}$$

The introduction of a stream function $\bar{\psi}$, defined by

$$u \equiv \frac{\partial \bar{\psi}}{\partial y} \quad \text{and} \quad v \equiv - \frac{\partial \bar{\psi}}{\partial x}$$

identically satisfies the continuity equation. Defining

$$\xi = \frac{x}{L}, \quad U = \frac{uL}{\alpha}, \quad Pr = \frac{\nu}{\alpha} \; ; \; V = \frac{vL}{\alpha}$$

$$\psi = \frac{\bar{\psi}}{\alpha} \; ; \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad Gr = \frac{g\,\beta(T_w - T_\infty)L^3}{\nu^2}$$

and introducing a coordinate transformation from (x,y) to streamline coordinates (ξ,ψ) the following relations are obtained.

$$\left.\frac{\partial}{\partial x}\right|_y = \frac{1}{L}\left[\left.\frac{\partial}{\partial \xi}\right|_\psi - V \left.\frac{\partial}{\partial \psi}\right|_\xi\right] \tag{4.7}$$

$$\left.\frac{\partial}{\partial y}\right|_x = \frac{U}{L} \left.\frac{\partial}{\partial \psi}\right|_\xi \tag{4.8}$$

The momentum and energy equations in this new coordinate system are

$$\frac{U}{Pr}\frac{\partial U}{\partial \xi} = U \frac{\partial}{\partial \psi}\left(U \frac{\partial U}{\partial \psi}\right) + Pr\ Gr\ \theta \tag{4.9}$$

$$\frac{\partial \theta}{\partial \xi} = \frac{\partial}{\partial \psi}\left[U \frac{\partial \theta}{\partial \psi}\right] \tag{4.10}$$

It is convenient to re-write the momentum equation as

$$\frac{1}{Pr}\frac{\partial U}{\partial \xi} - Ra \frac{\theta}{U} = \frac{\partial}{\partial \psi}\left(U \frac{\partial U}{\partial \psi}\right) \tag{4.11}$$

Equations (4.10) and (4.11) are of a form that allow the application of the variational procedure. A new variable, associated with the variational procedure of Equation (4.10) is defined as

$$Q(\xi,\psi) = \int_0^{\xi} U \frac{\partial\theta}{\partial\psi} d\xi \tag{4.12}$$

The significance of Q is clearer in (x,y) coordinates. Utilizing the relationship of Equation (4.8), Equation (4.12) may be re-written as

$$Q(x,y) = \int_0^{x} \frac{\partial\theta}{\partial y} dx$$

In this form it is apparent that Q is proportional to the heat flux through some section y between zero and x. Associating $\psi = 0$ with the wall, i.e., $y = 0$, then $Q(\xi,0)$ is proportional to the total heat flux from the wall between zero and $\xi = \frac{x}{L}$.

A new variable, associated with the variational procedure for Equation (4.11) is defined as

$$H(\xi,\psi) = \int_0^{\xi} U \frac{\partial U}{\partial\psi} d\xi \tag{4.13}$$

In (x,y) coordinates, H may be re-written as

$$H(x,y) = \int_0^{x} \frac{\partial U}{\partial y} dx$$

In this form, H is seen to be proportional to the shear force at some section y between zero and x. At the wall, i.e., $\psi = 0$, $H(\xi,0)$ is proportional to the total wall shear between zero and $\xi = x/L$.

The definition of the two error functions

$$E_1 = \int_0^{\xi} \int_0^{\psi_T} \left[\frac{\partial Q}{\partial \xi} - U \frac{\partial \theta}{\partial \psi}\right]^2 d\psi \; d\xi \tag{4.14}$$

$$E_2 = \int_0^{\xi} \int_0^{\psi_M} \left[\frac{\partial H}{\partial \xi} - U \frac{\partial U}{\partial \psi}\right]^2 d\psi \; d\xi \tag{4.15}$$

allows the writing of the variational equations for this coupled problem. In Equations (4.14) and (4.15) ψ_T represents a thermal 'penetration thickness' in terms of the stream function, while ψ_M represents a hydrodynamic 'penetration thickness'. Defining the $p_n(\xi)$ as the unknown functions associated with E_1 (and Q) and the $q_n(\xi)$ as the unknown functions associated with E_2 (and H), the variational equations are

$$\int_0^{\psi_T} \left(\frac{\partial Q}{\partial P_n}\right) \left(\frac{\partial Q}{\partial \xi} - U \frac{\partial \theta}{\partial \psi}\right) d\psi = 0 \tag{4.16}$$

$$\int_0^{\psi_M} \left(\frac{\partial H}{\partial q_n}\right) \left(\frac{\partial H}{\partial \xi} - U \frac{\partial U}{\partial \psi}\right) d\psi = 0 \tag{4.17}$$

For this problem it is assumed that the temperature profile can be adequately represented by a single unknown function $\psi_T(\xi)$. Likewise the velocity profile is assumed to be representable by a single unknown function $\psi_M(\xi)$. Under these assumptions, Equations (4.16) and (4.17) become

$$\int_{0}^{\psi_T} \left(\frac{\partial Q}{\partial \psi}_T\right) \left(\frac{\partial Q}{\partial \xi} - U \frac{\partial \Theta}{\partial \psi}\right) d\psi = 0 \tag{4.18}$$

$$\int_{0}^{\psi_M} \left(\frac{\partial H}{\partial \psi}_M\right) \left(\frac{\partial H}{\partial \xi} - U \frac{\partial U}{\partial \psi}\right) d\psi = 0 \tag{4.19}$$

From the definition of Q in Equation (4.12) and the energy equation (Equation (4.10)) it follows that

$$\frac{\partial Q}{\partial \psi} = \int_{0}^{\xi} \frac{\partial}{\partial \psi}\left(U \frac{\partial \Theta}{\partial \psi}\right) d\xi = \int_{0}^{\xi} \frac{\partial \Theta}{\partial \xi} d\xi$$

$$\frac{\partial Q}{\partial \psi} = \Theta \tag{4.20}$$

where the condition $\Theta = 0$ for $\xi = 0$ has been employed. Integrating yields

$$Q = \int_{\psi_T}^{\psi} \Theta \, d\psi \tag{4.21}$$

in which the constant of integration has been evaluated (as zero) by noting that, from the interpretation given to Q subsequent to Equation (4.12), Q must be zero for $\psi = \psi_T$. Then

$$\frac{\partial Q}{\partial \xi} = \left[\int_{\psi_T}^{\psi} \frac{\partial \Theta}{\partial \psi_T} d\psi - \Theta\Big|_{\psi=\psi_T}\right] \frac{d\psi_T}{d\xi} = \dot{\psi}_T \int_{\psi_T}^{\psi} \frac{\partial \Theta}{\partial \psi_T} d\psi = \frac{\partial Q}{\partial \psi_T} \dot{\psi}_T \tag{4.22}$$

since

$$\Theta_{\psi=\psi_T} = 0 .$$

Thus, Equation (4.18), the variational equation for the thermal problem may be written specifically as

$$\int_0^{\psi_T} \left[\int_{\psi_T}^{\psi} \frac{\partial \theta}{\partial \psi_T} \, d\psi \right] \left[\dot{\psi}_T \int_{\psi_T}^{\psi} \frac{\partial \theta}{\partial \psi_T} \, d\psi - U \frac{\partial \theta}{\partial \psi} \right] d\psi = 0 \tag{4.23}$$

With velocity and temperature profiles, Equation (4.23) represents one first order ordinary differential equation involving ψ_T and ψ_M.

From the definition of H in Equation (4.13) and the momentum equation (Equation 4.11) it follows that

$$\frac{\partial H}{\partial \psi} = \int_0^{\xi} \frac{\partial}{\partial \psi} \left(U \frac{\partial U}{\partial \psi} \right) d\xi = \frac{1}{Pr} \int_0^{\xi} \frac{\partial U}{\partial \xi} \, d\xi - Ra \int_0^{\xi} \frac{\theta}{U} \, d\xi \tag{4.24}$$

$$= \frac{U}{Pr} - Ra \int_0^{\xi} \frac{\theta}{U} \, d\xi$$

where the condition $U = 0$ for $\xi = 0$ has been employed. Integrating yields

$$H = \frac{1}{Pr} \int_{\psi_M}^{\psi} U d\psi - Ra \int_{\psi_T}^{\psi} \left(\int_0^{\xi} \frac{\theta}{U} \, d\xi \right) d\psi \tag{4.25}$$

in which the constant of integration has been evaluated (as zero) by noting that H must be zero for $\psi = \psi_M$. The lower limit of the second integral has been set at ψ_T because it is explicitly assumed that $\psi_T \leqslant \psi_M$ and therefore, since $\theta = 0$ for $\psi \geqslant \psi_T$, $\int_0^{\xi} \frac{\theta}{U} \, d\xi = 0$, $\psi_T \leq \psi \leqslant \psi_M$.

From Equation (4.25)

$$\frac{\partial H}{\partial \xi} = \frac{1}{Pr}\left[\int_{\psi_M}^{\psi} \frac{\partial U}{\partial \psi_M}\, d\psi - U\Big|_{\psi=\psi_M}\right]\dot{\psi}_M$$

$$- Ra\left[\int_{\psi_T}^{\psi} \frac{\theta}{U}\, d\psi - \left[\int_0^{\xi} \frac{\theta}{U}\, d\xi\right]_{\psi=\psi_T} \dot{\psi}_T\right]$$

but, $U\Big|_{\psi=\psi_M} = 0$

and $\left[\int_0^{\xi} \frac{\theta}{U}\, d\xi\right]_{\psi=\psi_T} = 0$

Therefore

$$\frac{\partial H}{\partial \xi} = \frac{\dot{\psi}_M}{Pr}\int_{\psi_M}^{\psi} \frac{\partial U}{\partial \psi_M}\, d\psi - Ra\int_{\psi_T}^{\psi} \frac{\theta}{U}\, d\psi \tag{4.26}$$

and

$$\frac{\partial H}{\partial \psi_M} = \frac{1}{Pr}\int_{\psi_M}^{\psi} \frac{\partial U}{\partial \psi_M}\, d\psi \tag{4.27}$$

Equation (4.19), the variational equation for the fluid-dynamic problem may be written specifically as

$$\int_0^{\psi_M}\left[\frac{1}{pr}\int_{\psi_M}^{\psi} \frac{\partial U}{\partial \psi_M}\, d\psi\right]\left[\frac{\dot{\psi}_M}{Pr}\int_{\psi_M}^{\psi} \frac{\partial U}{\partial \psi_M}\, d\psi - Ra\int_{\psi_T}^{\psi} \frac{\theta}{U}\, d\psi - U\frac{\partial U}{\partial \psi}\right] d\psi = 0 \tag{4.28}$$

With velocity and temperature profiles, Equation (4.28) represents another first order ordinary differential equation involving ψ_T and ψ_M. In Equation (4.28) it should be noted that, consistent with what has been said before, the term $\int_{\psi_T}^{\psi} \frac{\theta}{U} d\psi$ does not exist if $\psi > \psi_T$. Therefore the upper limit on the outer integral must be changed from ψ_M to ψ_T when this term is met in the integration.

The usual heat transfer correlations may be obtained in the following manner. The local Nusselt number

$$Nu_x \equiv \frac{h_x x}{k} = \frac{-\left.\frac{\partial T}{\partial y}\right)_o x}{(T_w - T_\infty)} = -x \left.\frac{\partial \theta}{\partial y}\right)_{y=0}$$

$$= -\xi U \left.\frac{\partial \theta}{\partial \psi}\right)_{\psi=0}$$

$$= -\xi \left.\frac{\partial Q}{\partial \xi}\right)_{\psi=0} = -\xi \frac{\partial}{\partial \xi} \int_{\psi_T}^{0} \theta d\psi \tag{4.29}$$

where the relations defined by Equations (4.8), (4.12) and (4.21) have been employed.

The average Nusselt number

$$\overline{Nu}_L \equiv \frac{\bar{h}L}{k} = \frac{-\int_o^x \left.\frac{\partial T}{\partial y}\right)_o dx}{(T_w - T_\infty)} = -\int_o^{\xi} U \left.\frac{\partial \theta}{\partial \psi}\right)_{\psi=0} d\xi$$

$$= -Q(\xi, 0) = -\int_{\psi_T}^{0} \theta d\psi \tag{4.30}$$

It is important to compute heat transfer quantities in the manner above rather than from derivatives of the temperature profile.

In order to compare velocity and temperature profiles in the physical plane it is necessary to determine y as a function of ψ. From the definition of the stream function it follows that

$$\frac{y}{L} = \int_0^{\psi} \frac{d\psi}{U} \tag{4.31}$$

B. <u>Velocity and Temperature Profiles</u>

For the coordinate system employed the condition that shearing stress and conductive flux be non-zero and finite near the wall requires (see Equation 4.8) that the velocity and temperature vary to the order of $\psi^{1/2}$ in the vicinity of the wall. For this reason the velocity profile was assumed as a power series in terms of $(\psi/\psi_M)^{n/2}$, while the temperature profile was assumed as a power series in terms of $(\psi/\psi_T)^{n/2}$.

The temperature profile was made to satisfy the following conditions

$$\Theta = 0 \quad \text{for} \quad \psi = \psi_T$$

$$\Theta = 1 \quad \text{for} \quad \psi = 0$$

$$\frac{\partial\Theta}{\partial\psi} = 0 \quad \text{for} \quad \psi = \psi_T$$

while the velocity profile satisfies

$$U = 0 \quad \text{for} \quad \psi = 0 \quad \text{and} \quad \psi = \psi_M$$

$$0 = \left[U \frac{\partial}{\partial\psi}\left(U \frac{\partial U}{\partial\psi}\right)\right]_{\psi=0} + \text{Ra}$$

The last condition for the velocity profile is obtained by evaluating Equation (4.9) at the wall ($\psi = 0$). This procedure is necessary because the usual 'smoothness' condition at $\psi = \psi_M$ is satisfied identically as a result of Equation (4.8) and the zero value for U at $\psi = \psi_M$.

The velocity and temperature profiles are

$$\theta = \left[1 - \left(\frac{\psi}{\psi_T}\right)^{1/2}\right]^2 \tag{4.32}$$

$$U = \left(\frac{4\,Ra}{3}\right)^{1/3} \psi_M^{\;2/3} \left[\left(\frac{\psi}{\psi_M}\right)^{1/2} - \frac{\psi}{\psi_M}\right] \tag{4.33}$$

Introducing these into Equations (4.23) and (4.28) yields

$$\psi_T^{\;1/3}\, \dot{\psi}_T - (Ra)^{1/3} r^{4/3} F_1(r) = 0 \tag{4.34}$$

and

$$\psi_M^{\;1/3}\, \dot{\psi}_M - (Ra)^{1/3} (Pr)\, F_2(r) = 0 \tag{4.35}$$

where

$$F_1(r) = \left[\frac{33}{7} - \frac{15\,r^{1/2}}{8}\right]\left(\frac{4}{3}\right)^{1/3} r^{-3/2} \tag{4.36}$$

and

$$F_2(r) = \left(\frac{81}{187}\right)\left(\frac{4}{3}\right)^{1/3}\left[16\left(1 - \frac{1}{r^{1/2}}\right)^2 \ln\left(\frac{1}{1-r^{1/2}}\right)\right. \tag{4.37}$$

$$+ \frac{5}{168} r^3 + \frac{3}{35} r^{5/2} + \frac{3}{20} r^2 + \frac{3}{10} r^{3/2} - \frac{4}{3} r - \frac{16}{3} r^{1/2}$$

$$\left. + \frac{333}{14} - 16\, r^{-1/2}\right]$$

$$r = \psi_T / \psi_M$$

Equation (4.34) may be rewritten as

$$\psi_M^{1/3} \; \dot{\psi}_M \; - \; (Ra)^{1/3} \, F_1(r) \; = \; 0 \qquad (4.38)$$

where $\psi_T^{1/3} \; \dot{\psi}_T \; = \; r^{4/3} \; \psi_M^{1/3} \; \dot{\psi}_M$

has been utilized.

A comparison of Equations (4.35) and (4.38) yields

$$Pr \; = \; \frac{F_1(r)}{F_2(r)}$$

Integration of Equation (4.38) yields

$$\psi_M(\xi) \; = \; (Ra)^{1/4} \left\{\frac{4}{3} \, F_1(r)\right\}^{3/4} \xi^{3/4} \qquad (4.40)$$

and from Equation (4.29), (4.30), (4.32) and (4.40) the heat transfer results may be written as

$$\frac{Nu_x}{\left(\frac{Gr_x}{4}\right)^{1/4}} \; = \; \frac{Pr^{1/4} \; r \left(F_1(r)\right)^{3/4}}{2 \; (3)^{\;3/4}} \qquad (4.41)$$

and

$$\frac{Nu_L}{\left(\frac{Gr_L}{4}\right)^{1/4}} \; = \; \frac{2}{(3)^{7/4}} \; Pr^{1/4} \; r \left[F_1(r)\right]^{3/4} \qquad (4.42)$$

Utilizing Equations (4.31), (4.33) and (4.30), the relationship between y and ψ may be written as

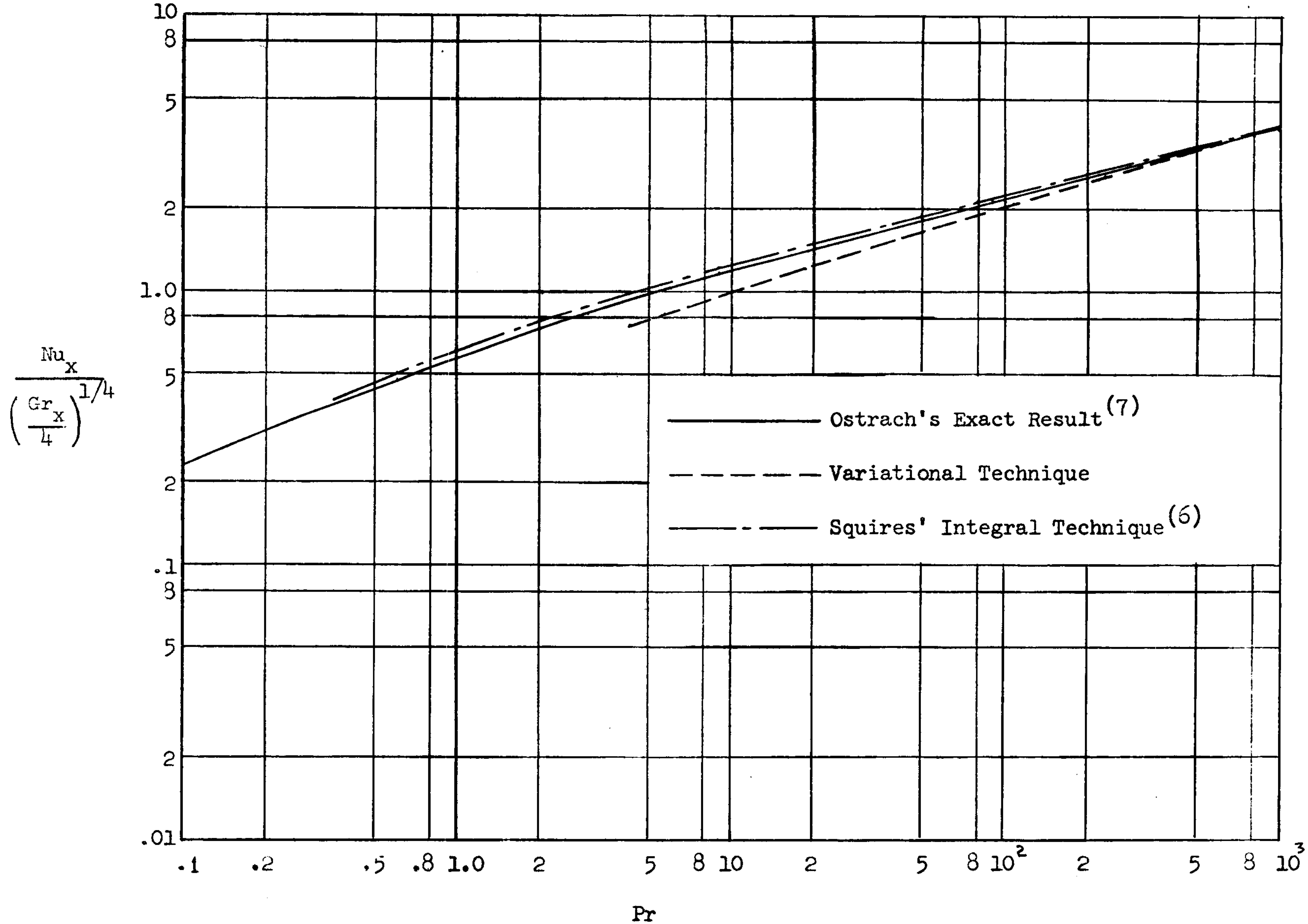
$$\frac{y}{x} \left\{ \frac{Gr_x}{4} \right\}^{1/4} \; = \; (2)^{1/3}(3)^{1/12} \frac{\left[F_1(r)\right]^{1/4}}{(Pr)^{1/4}} \, \ell n \left[\frac{1}{1 - \left(\frac{\psi}{\psi_M}\right)^{1/2}} \right] \qquad (4.43)$$

The assumption that $\psi_M \geqslant \psi_T$, explicit in the previous development, results, from consideration of Equation (4.39), in a minimum Prandtl number of 3.90 for this solution.

Calculations were made to compare this variational solution to previous solutions. A graphical representation of the heat transfer results, as well as the velocity and temperature profiles, for the several solutions is given in the figures which follow.

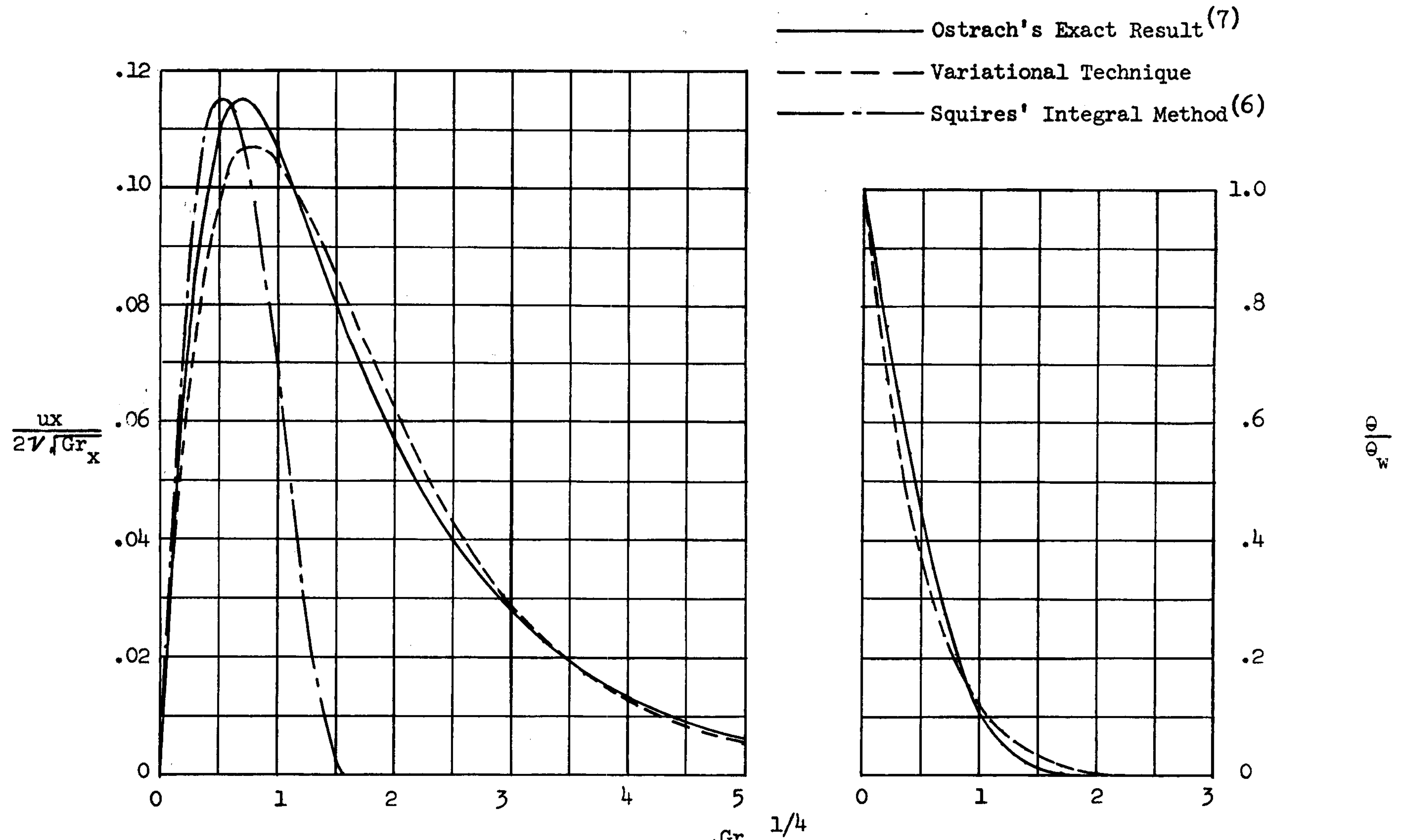
An examination of these figures reveals that the heat transfer results are poor except for high Prandtl numbers and that the velocity and temperature representation is not significantly better than that obtained from an integral approximate solution. It is possible that the relatively poor results obtained with this problem are more a measure of the unsuitableness of the coordinate system employed than of the variational procedure.

Heat Transfer Result - Isothermal Flat Plate



Comparison of Heat Transfer Result

Pr = 10

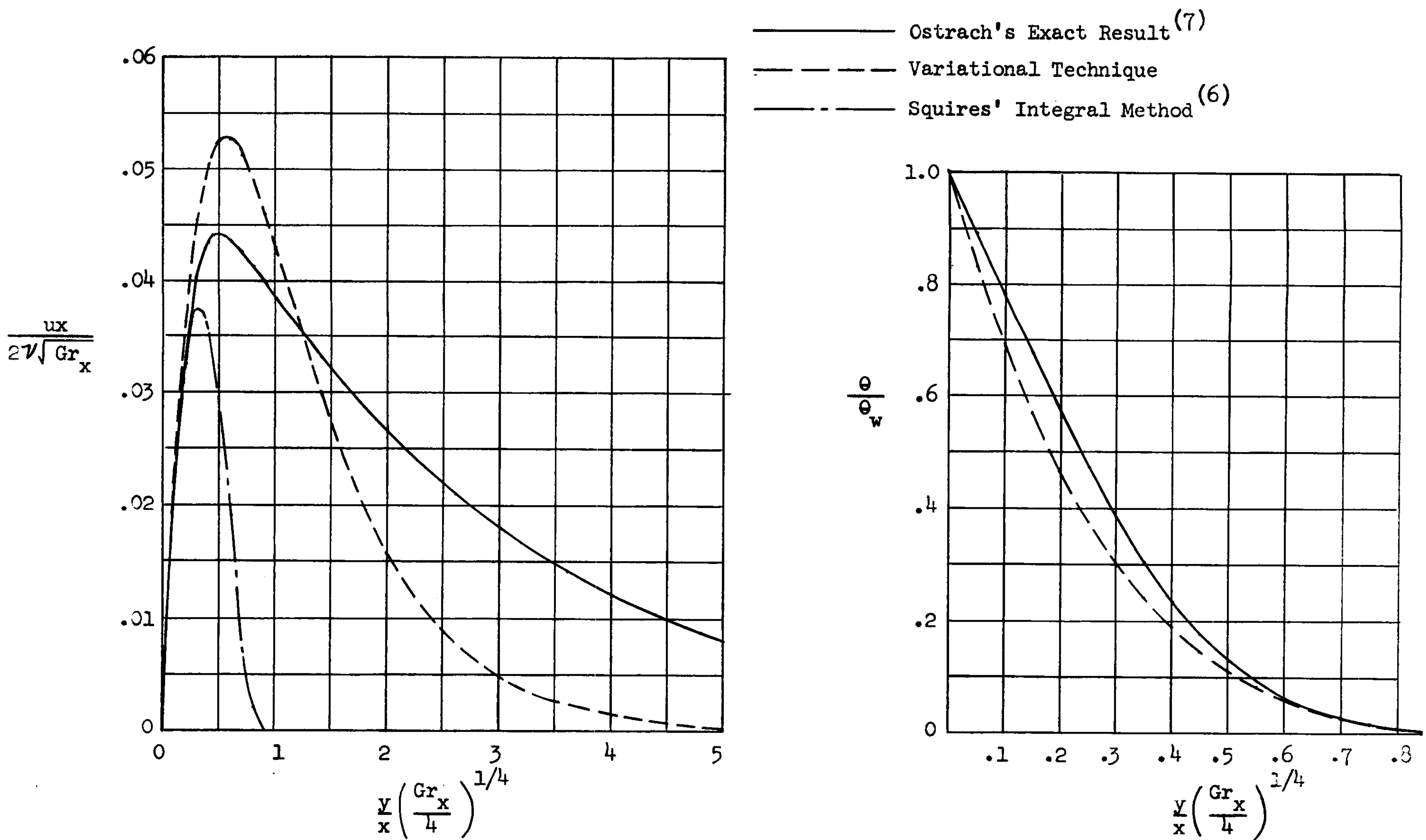


Comparison of Velocity Profile

Comparison of Temp. Profile

Comparison of Velocity Profile and Temperature Profile

Pr = 100



V. CONCLUSION

The present report clearly demonstrates that the variational solutions obtained have the accuracy desirable of approximate solutions. There exist a large number of problems to which this variational technique is apparently applicable. Work is now under way to test the technique for more complicated problems, and results of this continuing investigation will be the subject of future technical reports.

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